Solving $aE\overline{E} + b\overline{E} + c = 0$

We want to determine, for which triples $(a, b, c) \in \mathbb{C}^3$, the equation

$$a\,E\overline{E} \;+\; b\,\overline{E} \;+\; c \;=\; 0$$

admits a solution $E \in \mathbb{C}$.

The case $a = 0$

When $a = 0$, the equation reduces to

$$b\,\overline{E} + c = 0.$$

$$\text{If } b \neq 0, \qquad \overline{E} = -\frac{c}{b} \quad \Longrightarrow \quad E = -\frac{\overline{c}}{b}.$$

So a solution always exists, for every $c \in \mathbb{C}$.

If $b = 0$, then the equation is simply $c = 0$.

- If $b = 0$ and $c = 0$: every E is a solution.
- If $b = 0$ and $c \neq 0$: no solution exists.

Thus for $a = 0$, a solution exists unless

$$(b, c) = (0, k) \text{ for } k \neq 0.$$

From now on, assume $a \neq 0$

We divide the original equation by a and define

$$b' = \frac{b}{a}, \qquad c' = \frac{c}{a},$$

so the equation becomes

$$E\overline{E} + b'\,\overline{E} + c' = 0.$$

Rewrite as
$$c' = -\,E\overline{E} + b'\,\overline{E}.$$

Let us decompose E into magnitude and direction:

$$E = E_{\text{size}}\,E_{\text{dir}}, \qquad |E_{\text{dir}}| = 1.$$

Then
$$c' = -\,E_{\text{size}}^2 \; + \; b' E_{\text{size}}\,\overline{E_{\text{dir}}}.$$

For fixed E_{size}, as we vary E_{dir} over the unit circle, the term $b' E_{\text{size}}\,\overline{E_{\text{dir}}}$ traces out a circle of radius $|b'|E_{\text{size}}$ centered at the origin. Write $b_{\text{size}} = |b'|$. Then c' traces out a circle of radius $b_{\text{size}}E_{\text{size}}$ centered at $-E_{\text{size}}^2$, which lies on the real axis. Writing $c' = c_x + ic_y$, we have

$$(c_x + E_{\text{size}}^2)^2 + c_y^2 = (b_{\text{size}}E_{\text{size}})^2.$$

Finding the envelope of these circles

Define
$$F(c_x, c_y, z) = (c_x + z^2)^2 + c_y^2 - (b_{\text{size}}z)^2.$$

The envelope of the family is obtained from

$$F = 0, \qquad \frac{\partial F}{\partial z} = 0.$$

We have

$$\frac{\partial F}{\partial z} = 2(c_x + z^2)(2z) - 2b_{\text{size}}^2\, z = 2z\big(2(c_x + z^2) - b_{\text{size}}^2\big),$$

so the envelope occurs where either $z = 0$ or $2(c_x + z^2) = b_{\text{size}}^2$.

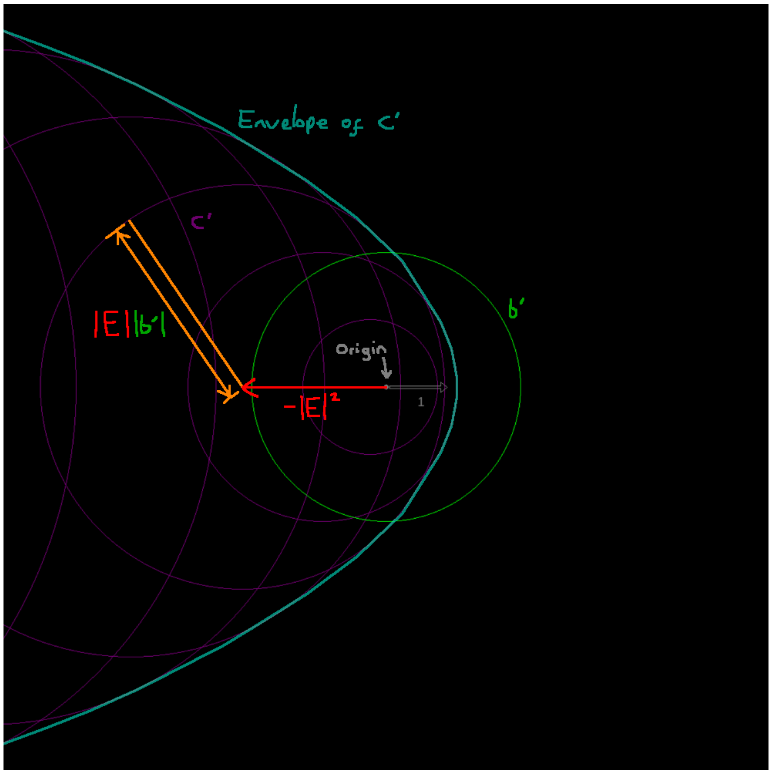
For $z = 0$: substituting into $F = 0$ gives $c_x^2 + c_y^2 = 0$, so $(c_x, c_y) = (0, 0)$.

For $2(c_x + z^2) = b_{\text{size}}^2$: we have $z^2 = \frac{b_{\text{size}}^2}{2} - c_x$. Substituting into $F = 0$ gives

$$\left(\frac{b_{\text{size}}^2}{2}\right)^2 + c_y^2 = b_{\text{size}}^2\left(\frac{b_{\text{size}}^2}{2} - c_x\right),$$

which simplifies to

$$c_y^2 = \frac{b_{\text{size}}^4}{4} - b_{\text{size}}^2 c_x.$$



Since z must be real, we require $z^2 = \frac{b_{\text{size}}^2}{2} - c_x \geq 0$, giving the constraint

$$c_x \leq \frac{b_{\text{size}}^2}{2}.$$

> The set of c' for which a solution E exists is the interior of this envelope.

Translating back to $c = ac'$ determines all triples (a, b, c) for which a solution exists.